Eight Flavors of the Axiom of Choice

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Abstract

The axiom of choice is likely the most discussed axiom in all of mathematics. Shown independent from the standard Zermelo-Frankel axioms of set theory by Gödel (1938) and Cohen (1963), with respect to the aforementioned ZF axioms, the axiom of choice has a plethora of equivalent statements. Remarkably, the intuitive truth value of many of these statements contradicts each other, despite all statements being logically equivalent. In this talk, we prove the equivalence of many formulations of the axiom of choice, including Zorn's Lemma, the Well-Ordering Principle and Trichotomy (with respect to set cardinality)

Quotations

The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma? –Jerry Bona

The Axiom of Choice is necessary to select a set from an infinite number of pairs of socks, but not an infinite number of pairs of shoes. –Bertrand Russell

The axiom gets its name not because mathematicians prefer it to other axioms. –A. K. Dewdney

Ordinals: Informally

Extremely roughly speaking, an ordinal is a generalization of the notion of a natural number. Any ordinal is defined as the set of everything less than it. For example, nothing is less than 0, so $0 = \emptyset$. Only 0 is less than 1, so $1 = \{\emptyset\}$. Continuing in this way, $n = \{1, 2, ..., n-1\}$. However, you can get bigger than just the naturals by constructing a limit ordinal, an ordinal that consists of an infinite number of things smaller than it $\omega = \{1, 2, 3, ...\}$. After you have ω, you can define $\omega + 1, \omega + 2, ...$ then you can take another limit ordinal and construct 2ω. You can keep doings this until you can make ω^2, etc.

Choice by Any Other Name

Theorem. *The following are equivalent (over ZF):*

(1) Axiom of Choice: for every set $\emptyset \notin X$, there exists $f : X \to \cup_{A \in X} A$ such that $\forall A \in X : f(A) \in A$

(2) Zorn's Lemma: Every non-empty partially order set such that every chain (totally ordered subset) has an upper bound contains a maximal element.

(3) Well-ordering principle: every set can be well-ordered.

(4) Tarski's Theorem about Choice: for every infinite set A, $|A| = |A \times A|$

(5) Every connected graph has a spanning tree

(6) Trichotomy: For any two sets A, B, either $|A| < |B|, |A| = |B|$ or $|A| > |B|$

(7) Every non-empty set can be equipped with group structure

(8) Cartesian Product of any family of non-empty sets is non-empty

We will show the following implications:

1. (1) $\implies$ (2)
2. (2) $\implies$ (3)

3. (3) $\implies$ (1)
4. (3) $\iff$ (4)
5. (2) $\implies$ (5) $\implies$ (1)
6. (3) $\iff$ (6)
7. (4) $\implies$ (7) $\implies$ (3)
8. (8) $\iff$ (1)

1 (1) $\implies$ (2)

Theorem. *The axiom of choice implies Zorn's Lemma.*

Proof. We prove this by contradiction using transfinite induction. Let $(X, \leq)$ be a partially ordered set such that every chain has an upper bound. Suppose that X does not contain a maximal element. Consider the map

$$s_< : X \mapsto \mathcal{P}(X)$$
$$x \mapsto \{y \in X : x < y\}$$

Since X does not contain a maximal element, $s_<(x)$ is non-empty for every element of X. Let c be a choice function on $\mathcal{P}(X)$ and consider $f = c \circ s_<$. By construction, this function satisfies the property that $\forall x \in X : f(x) > x$.

Let $\mathcal{C}$ be the set of all chains in X. Using the axiom of choice, define the following map:

$$g : \mathcal{C} \to X$$
$$g(C) \text{ is an upper bound of } C$$

Let $p \in X$ be any element. Define the the map h on the set of ordinals:

$$h(0) = p$$
$$h(\alpha^+) = f(h(\alpha))$$
$$h(\lambda) = f(g(\{h(\alpha) | \alpha < \lambda\}))$$

By construction, h is defined on all ordinals. Notice that h is strictly increasing, and is thus injective. Letting ON be the 'set' of all ordinals, we thus have that h is an order preserving bijection between ON and $h(ON)$. However, ON is not a definable set, so this is a contradiction. Thus X must contain a maximal element. □

2 (2) $\implies$ (3)

Theorem. *Zorn's Lemma implies the Well-Ordering Principle*

Proof. Let $\mathcal{X}$ be a set. Let $P = \{(X, \leq_X) | X \subset \mathcal{X} \text{ and } \leq_X \text{ is a well-ordering of } X\}$. Define $\preceq$ on P with respect to continuation as follows:

$$(X, \leq_X) \preceq (Y, \leq_Y) \iff X \subset Y \bigwedge \leq_X |X = \leq_Y |X \bigwedge \forall y \in Y/X, x \in X : x \leq_Y y$$

Let C be a chain in $(P, \preceq)$. Consider $U = \bigcup_{X \in C} X$ ordered by $\leq_U = \bigcup_{X \in C} \leq_X$. We claim that this is a well-ordering. Let $x, y \in U, x \neq y$. By definition of $\preceq$, there exists some $X \in \mathcal{C}$ such that $x, y \in X$. Since $\leq_X$ is a well ordering, we thus have that $x \leq_X y \vee y \leq_X x$. By definition of $\preceq$, $\leq_U$ must inherit this ordering. Similarly, let Z be a non-empty subset of U. By definition, there must be $X \in C$ such that $X \cap Z \neq \emptyset$. By definition of $\preceq$, Z inherits the minimum element of X. Thus $(U, \leq_U)$ is a subset of X equipped with well-ordering and a member of P. This shows that every chain in $(P, \preceq)$ has an upper-bound. By Zorn's Lemma, P thus has a maximal element $(M, \leq_M)$.

Suppose that there is $x \in X, x \notin M$. Extend $\leq_M$ to x by setting x greater than every element in M, contradicting maximality. Thus $M = X$ and $\leq_M$ is a well-ordering on X. □

3 (3) $\implies$ (1)

Theorem. *The Well-Ordering Principle implies the axiom of choice.*

Proof. Let $\mathcal{X}$ be a set. Let $f : \mathcal{X} \to \bigcup_{X \in \mathcal{X}} X$ send $X \mapsto \min(X)$, guaranteed to exist by the Well-Ordering Principle □

4 (3) $\iff$ (4)

Anecdote: Tarski tried to publish his theory in Comptes Rendus, a prestigious French journal. Both Frécht and Lebesgue refused. Fréchet wrote that an implication between two well-known true propositions is not a new result. Lebesgue wrote that an implication between two false propositions interests nobody.

Theorem. *The Well-Ordering Principle implies for any infinite set A, $|A| = |A \times A|$*

Let α be an ordinal such that $|A| = |\alpha|$, guarateed to exist by the Well-Ordering Principle. Consider the ordering on $\alpha \times \alpha$:

$$(a, b) < (c, d) \iff (\max(a, b), a, b) <_{lex} (\max(c, d), c, d)$$

Inductively. each element $(a, b) \in \alpha \times \alpha$ has less than or equal to $|\alpha|$ elements less than it, implying that $|\alpha| = |\alpha \times \alpha|$, as desired.

Lemma (Hartog's Lemma)**.** *For any set A, there exists an ordinal α such that there is no injection $\alpha \hookrightarrow A$*

Proof. Let $X = \{\beta \text{ is an ordinal} | \beta \hookrightarrow A\}$. By the fact that compositions of injections are injections, X is a transitive set of ordinals, and is thus an ordinal. However, if there existed a map $X \hookrightarrow A$, then X would contain itself, a contradiction. Thus X is an ordinal that does not inject into A. □

Theorem. *If, for any infinite set A, $|A| = |A \times A|$, then the Well-Ordering Principle.*

Proof. Let X be an infinite set and let $\aleph$ be an ordinal that cannot inject into X, guaranteed to exist by Hartog's Lemma. WLOG $X \cap \aleph = \emptyset$. By assumption, there exists a bijection $f : X \cup \aleph \leftrightarrow X \cup \aleph \times X \cup \aleph$.

Suppose that for any $x \in X$, $\aleph \times \{x\} \subseteq f(X)$. This would immediately imply that $f|_X$ is a surjection of X onto $\aleph$, a contradiction (since that would make $f|_X^{-1}$ an injection from $\aleph$ to X). Thus, for every $x \in X$, the set $S_x = \{\alpha \in \aleph | f(\alpha) \in \aleph \times \{x\}\}$ is non-empty.

Define $g : X \to \aleph$, $g(x) = \min S_x$. Since f a bijection, for any $x, y \in X, x \neq y \implies S_x \neq S_y$. Thus g is an injection from X into $\aleph$. Since $\aleph$ can be well ordered, X can be well-ordered.

Since finite sets can clearly be well-ordered, this completes the proof. □

5 (2) $\implies$ (5) $\implies$ (1)

Theorem. *Zorn's Lemma implies every graph has a spanning tree.*

Proof. Let (V, E) be a graph. Consider $P = \{(G', V') | G' \subset G, V' \subset V|_{G'}, (G', V') \text{ is a spanning tree}\}$. Order P by inclusion and note that the union of any chain of spanning trees must itself be a spanning tree. By Zorn's Lemma, there is a maximal spanning tree (M, E_M). It is maximal, so it must be that $M = V$, completing the proof. □

Theorem. *If every graph has a spanning tree, then the axiom of choice holds.*

Proof. Let $\mathcal{X}$ be a set. Let $A = \bigcup_{X \in \mathcal{X}} X$. Let $V = \mathcal{X} \sqcup A$ (disjoint union). Let $E = \{(u, v) | u, v \in A \wedge u \neq v\} \bigcup \{(X, x) | X \in \mathcal{X}, x \in X\}$. Let $G = (V, E)$ be a graph. By assumption there is some spanning tree T. Pick some vertex $v \in V$. For every $X \in \mathcal{X}$, there is a unique path in T from X to v. By construction of G, this path passes through an element of X, giving a choice function on $\mathcal{X}$. □

6 (3) $\iff$ (6)

Theorem. *The Well-Ordering Principle implies Trichtomy.*

Proof. Let A, B be sets. By assumption A and B can be well-ordered, so there are ordinals α, β that are bijective to A, B respectively. By basic set theory, any set of ordinals can be well-ordered, so either $|\alpha| < |\beta|, |\alpha| = |\beta|$ or $|\alpha| > |\beta|$, completing the proof. □

Theorem. *Trichotomy implies the Well-Ordering Principle.*

Proof. We prove this by contrapositive. Let X be a set that cannot be ordered, i.e. it does not inject into any ordinal. By Hartog's Lemma, let α be an ordinal that does not inject into X. Thus X and α are incomparable, implying Trichotomy does not hold. □

7 (4) $\implies$ (7) $\implies$ (3)

Theorem. *If all infinite sets A satisfy $|A| = |A \times A|$, then all sets can be equipped with group structure.*

Proof. It is clear that if A is a finite set, it can simply be equipped with cyclic structure of order $|A|$.

Let A be an infinite set. Let F be the set of all finite subsets of A. By elementary group theory, F equipped with symmetric difference is a group.

We have that $F = \cup_{n \in \mathbb{N}} A^n$. Since $|A| = |A \times A|$, we have that $|F| = |\cup_{n \in \mathbb{N}} A^n| = |\mathbb{N} \times A| \leq |A \times A| = |A|$. Thus there exists a bijection between A and F. Since F is a group, A can inherit it's group structure, completing the proof. □

Theorem. *If all sets can be equipped with group structure, then all sets can be well-ordered.*

Proof. Let $\aleph$ be an ordinal that does not inject into A, whose existence is guaranteed by Hartog's Lemma. WLOG $\aleph \cap A = \emptyset$. Let $G = (A \cup \aleph, \cdot)$ be a group.

Suppose that there is a $a \in A$ such that for all $\beta \in \aleph$, we have that $a \cdot \beta \notin \aleph$. Thus $x \mapsto a \cdot x$ is an injection from $\aleph \hookrightarrow A$, a contradiction. Thus for all $a \in A$, $S_a = \{(\alpha, \beta) \in \aleph \times \aleph | a \cdot \alpha = \beta\}$ is non-empty.

Define $f : A \to \aleph \times \aleph$ such that $f(a) = \min S_a$, definable since $\aleph \times \aleph$ can be well-ordered lexicographically. This we have an injection from A to a well-ordered set, so A can thus be well-ordered. □

8 (8) $\iff$ (1)

Theorem. *The axiom of choice is equivalent to the statement that the Cartesian product of any family of non-empty sets is non-empty.*

Proof. By definition, the cartesian product of a family of sets X_i indexed by I is a family of maps:

$$\left\{ f : I \to \bigcup_{i \in I} X_i \middle| \forall i \in I : f(i) \in X_i \right\}$$

Clearly any element of this product is a choice function on $\{X_i\}_{i \in I}$ and any choice function on $\{X_i\}_{i \in I}$ is a member of this product. Noting that any set of sets indexes itself completes the proof. □