Fixed Point Theorems

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September 4, 2020

Abstract

The fixed point of some function f is a point such that $f(x) = x$. Fixed point theorems are theorems that guarantee that certain types of functions have fixed points. In this talk, we divide fixed points into three categories: topological, diagonal and iterative. We then prove the Brouwer, Roger and Banach fixed point theorems as representative examples of each of the categories.

1 Introduction

A fixed point of a function $f(x)$ is a point such that $f(x) = x$. Fixed point theorems are theorems that that show that various types of functions have fixed points. Sometimes, they even give you a method for finding those fixed points.

Fixed point theorems can roughly be divided into 3 categories: Topological, Diagonal, and Iterative.

Topological fixed points are non-constructive. If $f : \mathbb{R} \to \mathbb{R}$ is continuous a function such that $f(0) < 0$ and $f(1) > 1$, then it must be that $f(x)$ crosses x at some point, so it must have a fixed point. However, there is no principled way to select the fixed point (especially since it's possible that there are an uncountable number of fixed points). The Brouwer fixed point theorem shows the existence of topological fixed points (under certain conditions).

Diagonal fixed points are constructed by feeding a function into itself (sometimes called quining). This technique is very strong and proves, among other things, Gödel's Incompleteness, the Halting Problem, etc. Roger's fixed point theorem shows the existence of diagonal fixed points.

Iterative fixed points can be found by repeatedly iterating a function on any point. The function $f(x) = -\frac{x}{2}$ has a fixed point given by $\lim_{n\to\infty} f^n(x)$ for any point x. The Banach fixed point theorem shows the existence of iterative fixed points under certain conditions.

Why are fixed points interesting? They can be used in a bunch of ways to prove interesting things. Godel's incompletness theorems are about fixed points of sentences. The existence of mixed strategy Nash Equilibrium is a consequence of Brouwer Fixed Point Theorem. The existence and uniqueness of ODE's can be shown with Banach Fixed Point Theorem. Roger Fixed Point theorem shows the existence of computer programs that can "talk about themselves" in some sense

2 Iterative Fixed Points

Let (X, d) be a complete metric space. Metric space means you can measure stuff in a way that satisfies the triangle inequality. Complete means that it contains all it's limit points. $\mathbb{R}^n$ is the example you should have in mind. A map $T : X \to X$ is a *contraction mapping* if there exists $0 \leq q < 1$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

That is, the map T must shrink distances between points when you apply it. Notice that contraction mappings must be continuous because discontinuities violate the contraction property.

Theorem 2.1 (Banach Fixed Point Theorem)**.** *For any (continuous) contraction mapping T on a non-empty complete metric space, T has a unique fixed point equal to $\lim_{n\to\infty} T^n(x)$ for arbitrarily chosen x.*

Proof. Let X be our space and pick $x_0 \in X$ arbitrarily. Define a sequence $\{x_n\}$ such that $x_n = T^n(x_0)$. Inductively, we have that

$$d(x_{n+1}, x_n) \leq q^n d(x_1, x_0)$$

Furthermore, the sequence $\{x_n\}$ is Cauchy. Let m, n be such that $m > n$. Notice that since d satisfies the triangle inequality:

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)$$

T is a contraction mapping:

$$\leq q^{m-1} d(x_1, x_0) + q^{m-2} d(x_1, x_0) + ... + q^n d(x_1, x_0)$$

$$\leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k$$

$$\leq q^n d(x_1, x_0) \left(\frac{1}{1-q} \right)$$

Since $0 \leq q < 1$, picking sufficiently large N will guarantee that x_m, x_n are within ε for any $\varepsilon > 0$.

By completeness, $x_n \to x$ has a limit in X. By continuity of T, we can interchange limits, giving $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} T_{n-1}) = T(x)$

Uniqueness is a simple consequence of the face that $d(x, y) \neq d(T(x), T(y))$ for any $x \neq y$. □

3 Diagonal Fixed Points

Diagonal fixed points are the fixed points of logic and computers. First we have to define the domain of the functions that have fixed points. In this case, it's the set of all partial recursive functions from $\mathbb{N} \to \mathbb{N}$. Partial means that it might not be defined on some input (contrast to total, which means defined on every input). Recursive means that you can build it out of a small set of elementary functions along with some rules of construction. An easy way to think about partial recursive functions is that they are just computer programs. This isn't just a good abstraction; we can prove that a function is partial recursive if and only if you can write down a computer program that computes it.

How do we do this? On a high level, think of a computer program as just a set of transition rules between memory states of your computer, i.e. a bunch of statements of the sort "if we have a 1 in memory location 42 and a 0 in memory location 46, then at the next step, we must have a 1 in memory location 5". This means that we can define a mathematical function that takes a program and a memory state and returns the next memory state. Using this, we can define another function that asks if there is a sequence of memory states, each of which came from the one before it, such that the last state is the halting state with a particular output.

Now that we know what the domain is, how do we define a fixed point? First we fix some *admissible numbering* ϕ of partial recursive functions (also sometimes called an *effective enumeration*. This is just some sequence ϕ_n that contains every partial recursive function and that you can write down a computer program that prints ϕ_n in order. Why do we know that this exists? Well computer programs can be thought of as strings from some finite alphabet (ASCII if you want). We can just write a compute program that prints out all possible strings. Most of these won't be computer programs, but that's fine because they'll be the empty program, which is just the unique program that's undefined on every input.

Given this, a fixed point of a function F is an index e such that $\phi_e \simeq \phi_{F(e)}$, where $\phi_n \simeq \phi_m$ if ϕ_n, ϕ_m agree whenever either of them is defined (they are, in some sense, the same computer program). This leads us to our fixed point theorem.

Theorem 3.1 (Roger Fixed Point Theorem (also called Kleene's 2nd Recursion Theorem))**.** *Every total computable function F has a fixed point.*

Proof. First we'll have to define a total function h. Defining this is math is hard, so we will define it with programming

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def h(x):
    def f(y):
        e = \phi_x(x)
        return \phi_e(y)
    return f
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Since ϕ is an admissible numbering, this is a real computer program. What does this program even do? Well if $\phi_x(x)$ is defined, then $\phi_{h(x)}$ is the program f that we define in h with x fixed. Since $\phi_x(x)$ is defined, $f(y)$ will always return $\phi_{\phi_x(x)}$. Thus $\phi_{h(x)} \simeq \phi_{\phi_x(x)}$.

Now let F be any total computable function. Consider $F \circ h$. Since ϕ is an admissible numbering, every partial recursive function appears somewhere. Namely, there exists some e such that $\phi_e \simeq F \circ h$. By the above, we have that $\phi_{h(e)} \simeq \phi_{\phi_e(e)}$. However, $\phi_e \simeq F \circ h$, so we have that $\phi_e(e) = F(h(e))$. Thus $\phi_{\phi_e(e)} \simeq \phi_{F(h(e))}$. By transitivity, $\phi_{h(e)} \simeq \phi_{F(h(e))}$, so $h(e)$ is a fixed point of F. □

4 Topological Fixed Points

First we'll prove a combinatorial version of this fixed point theorem.

Lemma 4.1 (Sperner's Lemma). *Given an n-dimensional simplex and a triangulation T colored into $n+1$ colors such that:*

- *The $n+1$ vertexes that are the "corners" of T are colored differently.*
- *The vertexes on any k-dimensional surface of the large simplex are only colored the colors of the "corners" included in that subface.*

Then, T has an odd number of simplices that contain all colors. Crucially, there exists 1 such simplex.

What does this mean? Well, a 1-simplex is a line segment, so the claim is that if we put vertexes on a line and color the endpoints differently, the color must swap an odd number of times. Let's start by proving the one dimensional case.

Proof. If we put only 2 vertexes on the line, then it's obvious. Given that it's true for n vertexes, let's add one more vertex. If you add it between switching colors, then you must maintain the number of switches. If you add it between non-switching colors, you can either add 2 switches or maintain the same number. Odd numbers plus 2 are still odd, so that completes the proof. □

A 2-simplex is a triangle. Sperner's Lemma says that if you color a triangle made of smaller triangles into three colors with the corners being different colors and the edges between the corners only containing those two colors, then there exist an odd number of sub-triangles that have all 3 colors. Let's prove this (actually we're just going to prove it for all dimensions now, but whatever).

In general, think of a n-simplex as the set of points unit taxicab distance from the origin, i.e. $\Delta^n = \{P \in \mathbb{R}^{n+1} | \sum P_i = 1\}$

Proof. Suppose that we have Sperner's Lemma for n dimensions. Construct a graph G in the following way. Let V be the set of all the "spaces" in T plus one vertex that represents the outside (call this o). Pick a random color c and let two vertexes have an edge between then if they share a $n-1$ simplex face with every color present except for c. Add edges between vertexes and o if they "share" a face with the outside. Note that by definition of the coloring of T, only one face that is adjacent with the outside contains all but the color c. Apply Sperner's Lemma to this face to show that it must contain and odd number of $n-1$ simplices with all colors but c, which shows that o has odd degree.

Now observe that the vertexes that aren't o must have degree 0, 1, or 2. Why is this the case? Suppose that the degree is at least 1. That means that we know the colors of all but 1 vertex. Suppose that this vertex is colored c. Then any face that contains this vertex will not produce an edge, so the degree is 1. Suppose that this vertex is not colored c. Then it will be the same as some other vertex, so we can include a face that swaps out those two vertexes. However, every other face will contain both of these vertexes, so it cannot produce an edge. Thus the degree is 2. This also shows that a simplex contains every color if and only if the degree is 1.

By elementary graph theory, the sum of all degrees of all vertexes in G must be even. Since o has odd degree, there thus must be an odd number of vertexes with degree 1, completing the induction. □

Now we can prove Brouwer:

Theorem 4.2 (Brouwer Fixed Point Theorem). *Every continuous function from a convex set to itself has a fixed point.*

Proof. First we consider the case of a continuous function f from an n dimensional simplex Δ^n to itself. By definition, we have that

$$\Delta^n = \{P \in \mathbb{R}^{n+1} | \sum P_i = 1\}$$

For every point in $P \in \Delta^n$, we can note that $\sum_{f(P)_i} = 1$ also. By the Pigeon Hole principle, we for every point P, there exists an k such that $f(P)_k \leq P_k$. In particular, such a k can be selected such that $P_k \neq 0$.

Define the map $color : \Delta^n \to \{1, ..., n+1\}$ such that each $P \in \Delta^n$ maps to a k that satisfies the above (we use axiom of choice to do this). Let T be any triangulation of Δ^n colored according to $color$.

Notice that since we have that $P_{color(P)} \neq 0$, any "corner" will be colored exactly the index that it is non-zero. Furthermore, any point on some sub-simplex of Δ^n will only contain colors of the endpoints. Thus $color$ is a Sperner Coloring of T, allowing us to apply Sperner's Lemma, which guarantees the existence of a n-dimensional simplex with all $n+1$ colors.

By making T arbitrarily fine, we can select a sequence of n-dimensional simplicies that are all-chromatic and whose diameter goes to 0. Construct a sequenc e $\{p_i\}$ by picking 1 point from each simplex. Since Δ^n is compact, we can pick some convergent subsequence that converges to P. Suppose that there exists some index j such that $f(P)_j > P_j$. By continuity, any sufficiently small n-simplex around P will not have all $n+1$ colors, a contradiction. Thus we have that for every j, $f(P)_j \leq P_j$. Since $\sum f(P)_i = \sum P_i = 1$, this must be equality. Thus $f(P) = P$ and P is a fixed point.

To generalize to arbitrary convex sets, we simply use the fact that convex projection, the fact that taking a set and mapping to the closest point of some convex set, is well-defined and continuous. Given any convex set S and a function $g : S \to S$, construct a large n-simplex around it and compose g with the convex projection map. Any fixed point must lie in the range of g and thus must lie in S, implying it is a fixed point of g. □